Question 1	Question 2	Question 3	Question 4	Sum	Final score

Written exam ('terzo appello') of Teoria delle Funzioni 1 for Laurea Magistrale in Matematica - 28 June 2012.

SURNAME .... NAME .... MATRICOLA ....

PLEASE NOTE. During this exam, the use of notes, books, calculators, mobile phones and other electronic devices is strictly FORBIDDEN. Personal belongings (e.g., bags, coats etc.) have to be placed far from the seat: failure to do so will result in the annulment of the test. Students are entitled to use only a pen. The answers to the questions below have to be written in these pages. Drafts will NOT be considered. Marked tests will be handed out in room 1BC45 on 29 June 2012 at 10:00.

Duration: 150 minutes

Question 1.

(i) Give the definition of kernel of mollification and of mollifier $A_\delta f$ with step δ for a function $f : \Omega \to \mathbb{R}$, where Ω is an open set in $\mathbb{R}^N$.
(ii) Prove that there is no function $u \in L^1(\mathbb{R}^N)$ such that

$$u * f = f, \quad \forall\, f \in L^1(\mathbb{R}^N).$$

(Hint: use (i).)
(iii) Let $f \in L^\infty(\mathbb{R}^N)$. Is it true that $A_\delta f \to f$ in $L^\infty(\mathbb{R}^N)$ as $\delta \to 0$? (give a detailed motivation)

Answer:

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Question 2.

(i) Let Ω be an open set in $\mathbb{R}^N$ and $l \in \mathbb{N}$, $p \in [1, \infty]$. Give the definition of the Sobolev spaces $W^{l,p}(\Omega)$ and $\widetilde{W}^{l,p}(\Omega)$.

(ii) Let $f : (-\pi/4, \pi/4) \to \mathbb{R}$ be the function defined by

$$f(x) = \frac{1}{\log|\sin x|}\,,$$

for all $x \in (-\pi/4, \pi/4) \setminus \{0\}$ and $f(0) = 0$. Prove that the weak derivative f'_w of f exists in $(-\pi/4, \pi/4)$ and compute it.

(iii) Find all values of $p \in [1, \infty]$ such that the function f defined above belongs to $W^{1,p}(-\pi/4, \pi/4)$.

Answer:

Question 3.

(i) State Gagliardo's inequality in the general case, specifying the range of exponents p for which it is valid.

(ii) Does Gagliardo's inequality hold for all functions $f \in W^{1,p}(\Omega)$ where Ω is a bounded domain in $\mathbb{R}^N$?

(iii) State the Hardy-Littlewood-Sobolev inequality.

Answer:

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Question 4.

(i) Give the definition of compact operator and state the Rellich-Kondrakov Theorem.

(ii) State the Trace Theorem for the Sobolev Space $W^{l,p}(\Omega)$ in terms of the appropriate Besov spaces.

(iii) Assume that Ω is a bounded open set for which the Trace Theorem holds. Let $f \in W^{1,p}(\Omega)$. Prove that if the function $f - 30$ belongs to $W_0^{1,p}(\Omega)$ then $\mathrm{Tr} f = 30$.